Supplement

- The singularities at $t = \pm 1$ come from
  $\operatorname{Ker} D^+_{A,t} \neq 0 \quad t = \pm 1$

- Higher rank case $\quad \Phi \sim i \begin{bmatrix} \mu_1 & & & 0 \\ & \mu_2 & & \\ & & \ddots & \\ 0 & & & \mu_r \end{bmatrix} + \cdots$

$$\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_r \qquad \sum \mu_\alpha = 0 \qquad \circ - \circ - \circ - \circ \cdots \circ - \circ$$

$\Longrightarrow$ solution of Nahm's equation on $(\mu_1, \mu_r) \setminus \{\mu_2, \mu_3 \cdots \mu_{r-1}\}$

rk of v.b'dle may change on each interval

☆ This result has a rigorous proof only when all μ_i : distinct

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Today : Solve Nahm's equation

$T_0, T_1, T_2, T_3 \qquad \dfrac{\nabla}{ds} T_i + \dfrac{1}{2} \sum \varepsilon_{ijk} [T_j, T_k] = 0 \qquad i = 1, 2, 3$

gauge transform : $u : [-1, 1] \longrightarrow U(k)$

$$T_i^u = u^{-1} T_i u \qquad (i = 1, 2, 3)$$

$$T_0^u = u^{-1} T_0 u + u^{-1} \frac{du}{ds}$$

Choose $\mathbb{H} \cong \mathbb{C}^2 \qquad$ Put $\begin{cases} \alpha := \frac{1}{2}(T_0 + i T_1) \\ \beta := \frac{1}{2}(T_2 + i T_3) \end{cases}$

Nahm's equ. $\Longleftrightarrow$

$$\begin{cases} \dfrac{d\beta}{ds} + 2[\alpha, \beta] = 0 & \text{cpx equation} \\ \dfrac{d}{ds}(\alpha + \alpha^*) + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0 & \text{real equation} \end{cases}$$

check: $\alpha+\alpha^* = iT_1$

$$2[\alpha, \alpha^*] = -\tfrac{1}{2}[T_0+iT_1, T_0-iT_1] = i[T_0, T_1]$$

$$2[\beta, \beta^*] = \qquad = i[T_2, T_3]$$

Observation

1. The cpx equation is invariant under the cpx gauge transform :

$g : [-1,1] \longrightarrow GL(\mathbb{C})$ $\qquad \begin{cases} \alpha^g = g^{-1}\alpha g + \frac{1}{2} g^{-1}\frac{d}{ds} g \\ \beta^g = g^{-1}\beta g \end{cases}$

$$\Longrightarrow \frac{d\beta^g}{ds} + 2[\alpha^g, \beta^g] = g^{-1}\left(\frac{d\beta}{ds} + 2[\alpha, \beta]\right) g$$

2. The cpx equation is locally trivial
if ignore the bdry condition

We solve $\alpha^g = g^{-1}\alpha g + \frac{1}{2} g^{-1}\frac{d}{ds} g = 0$.

Then $\dfrac{d\beta^g}{ds} + 2[\alpha^g, \beta^g] = 0 \iff \beta^g = \text{const}$

$\therefore$ General solutions of cpx equation are

$$\alpha = -\frac{1}{2}\frac{d}{ds} g \cdot g^{-1} , \quad \beta = g \beta_0 g^{-1}$$

β_0 : constant $\in \mathfrak{gl}(\mathbb{C})$, g : $GL(\mathbb{C})$-valued

Th (Donaldson, CMP 96 (1984))
Suppose a solution of the cpx equation is given.
$\Rightarrow$ Then $\exists$ cpx gauge transform g s.t.
(α^g, β^g) satisfies the real equation.
Moreover it is unique up to a (unitary) gauge transform.

"Dolbeault" operators

$$\bar\partial_\alpha := \frac{1}{2}\frac{d}{ds} + \alpha \quad , \quad \partial_\alpha := \frac{1}{2}\frac{d}{ds} - \alpha^*$$

$$\bar\partial_\beta := \beta \quad , \quad \partial_\beta := -\beta^*$$

- cpx equ $\Leftrightarrow$ $[\bar\partial_\alpha, \bar\partial_\beta] = 0$ integrability
- cpx gauge transfrm : $\begin{cases} \bar\partial_{\alpha^g} = g^{-1}\bar\partial_\alpha g \\ \bar\partial_{\beta^g} = g^{-1}\bar\partial_\beta g \end{cases}$

$\hat{F}(\alpha,\beta) :=$ LHS of real equation : (1,1)-part of the curvature

$$= 2\left([\partial_\alpha, \bar\partial_\alpha] + [\partial_\beta, \bar\partial_\beta]\right)$$

We kill the ambiguity of unitary gauge transfrms by introducing:

$h := g^{*-1}g^{-1}$ $\quad h^{-1} = gg^*$ $\quad (\Rightarrow h^* = h \ , \ h > 0)$

Formulas

(1) $\partial_{\beta^g} = g^{-1}(\partial_\beta + h^{-1}\partial_\beta h)g$

$\because$ LHS $= -\beta^{g*} = -(g^{-1}\beta g)^* = -g^*\beta^* g^{*-1}$

RHS $= g^{-1}\Big(-\beta^* + gg^*(\underbrace{-[\beta^*, g^{*-1}g^{-1}]}_{g^{*-1}g^{-1}\beta^* - \beta^* g^{*-1}g^{-1}})\Big)g$

$= -g^*\beta^* g^{*-1}$ //

(2) $\partial_{\alpha^g} = g^{-1}(\partial_\alpha + h^{-1}\partial_\alpha h)g$

$\because$ LHS $= \frac{1}{2}\frac{d}{ds} - \alpha^{g*} = \frac{1}{2}\frac{d}{ds} - \left(g^{-1}\alpha g + \frac{1}{2}g^{-1}\frac{dg}{ds}\right)^*$

RHS $= g^{-1}\left(\frac{1}{2}\frac{d}{ds} - \alpha^* - \frac{1}{2}\frac{d}{ds}(gg^*)g^{*-1}g^{-1} - gg^*[\alpha^*, g^{*-1}g^{-1}]\right)g$ //

(3) $\hat{F}(\alpha^g, \beta^g) = g^{-1}\left(\hat{F}(\alpha,\beta) - 2\{\bar{\partial}_\alpha(h^{-1}\partial_\alpha h) + \bar{\partial}_\beta(h^{-1}\partial_\beta h)\}\right)g$

$\because$

$$\frac{1}{2}\hat{F}(\alpha,\beta) = [\partial_\alpha, \bar{\partial}_\alpha] + [\partial_\beta, \bar{\partial}_\beta]$$

$$\frac{1}{2}\hat{F}(\alpha^g,\beta^g) = [\partial_{\alpha^g}, \bar{\partial}_{\alpha^g}] + [\partial_{\beta^g}, \bar{\partial}_{\beta^g}]$$

$$= g^{-1}\Big(\underbrace{[\partial_\alpha + h^{-1}\partial_\alpha h, \bar{\partial}_\alpha] + [\partial_\beta + h^{-1}\partial_\beta h, \bar{\partial}_\beta]}_{\frac{1}{2}\hat{F}(\alpha,\beta)} \Big) g$$

//

$I \subset (-1,1)$: closed interval

Lemma. (α, β) : fix

Let $\mathcal{L}(g) := \frac{1}{2} \int_I (|\alpha^g + \alpha^{g*}|^2 + 2|\beta^g|^2)\, ds$

$(\mathcal{L} : \{ g \in C^1(I, GL(\mathbb{R})) \mid g|_{\partial I} = 1 \} \to \mathbb{R}_{\geq 0})$

$\Longrightarrow$ $\hat{F}(\alpha^g, \beta^g) = 0$ is the Euler-Lagrange equation of $\mathcal{L}$.

(proof) May assume $g = 1$ & $(\delta g)^* = \delta g$ ($\Leftarrow \mathcal{L}$ is inv. under unitary gauge)

$$\delta \mathcal{L} = \mathrm{Re} \int_I \mathrm{Tr}\left[(\alpha + \alpha^*) \delta(\alpha + \alpha^*) + 2\beta \delta\beta^* \right] ds$$

$$\delta\alpha = [\alpha, \delta g] + \frac{1}{2} \frac{d}{ds}(\delta g), \quad \delta\beta = [\beta, \delta g]$$

$$\therefore \delta(\alpha + \alpha^*) = [\alpha - \alpha^*, \delta g] + \frac{d}{ds}(\delta g)$$

$$\therefore \delta \mathcal{L} = \mathrm{Re} \int_I \mathrm{Tr}\Big((\alpha + \alpha^*) \big(\frac{d}{ds}\delta g + [\alpha - \alpha^*, \delta g] \big) + 2\beta [\delta g, \beta^*] \Big)$$

$$= \mathrm{Re} \int_I \delta g \Big(\underbrace{-\frac{d}{ds}(\alpha + \alpha^*) - 2[\alpha, \alpha^*] - 2[\beta, \beta^*]}_{= -\hat{F}(\alpha, \beta)} \Big)$$

integration by parts

//

Assume $\alpha = 0$, $\beta = \text{const}$

$$\Rightarrow \mathcal{L}(g) = \frac{1}{8}\int_I \left(\left| g^{-1}\frac{dg}{ds} + \frac{dg^*}{ds} g^{*-1} \right|^2 + 2\left| g^{-1}\beta g \right|^2 \right)$$

$$= \frac{1}{8}\,\mathrm{Re}\int_I \mathrm{Tr}\left(\left(g^{-1}\frac{dg}{ds} + \frac{dg^*}{ds} g^{*-1} \right)^2 + 2 g^{-1}\beta g g^* \beta^* g^{-1*} \right)$$

$$= \frac{1}{8}\int_I \mathrm{Tr}\left(h^{-1}\frac{dh}{ds} \right)^2 + 2\,\mathrm{Tr}(\beta h^{-1}\beta^* h)$$

$\because$ $\mathrm{Tr}(g^{-1}\beta g g^* \beta^* g^{-1*}) = \mathrm{Tr}(\beta h^{-1}\beta^* h)$

$$h^{-1}\frac{dh}{ds} = g g^* \frac{d(g^{*-1}g^{-1})}{ds}$$

$$= -\left(\frac{dg}{ds} g^* + g\frac{dg^*}{ds} \right) g^{*-1} g^{-1} = -\left(\frac{dg}{ds} g^{-1} + g\frac{dg^*}{ds} g^{*-1} g^{-1} \right)$$

$$\therefore \left(h^{-1}\frac{dh}{ds} \right)^2 = \left(\frac{dg}{ds} g^{-1} \right)^2 + g\left(\frac{dg^*}{ds} g^{*-1} \right)^2 g^{-1}$$

$$+ \frac{dg}{ds}\frac{dg^*}{ds} g^{*-1} g^{-1} + g\frac{dg^*}{ds} g^{*-1} g^{-1}\frac{dg}{ds} g^{-1}$$

Taking Tr, we get $\uparrow$ //

<u>Remark.</u>

$$\mathcal{H} := \{ h \mid h = h^*,\ h > 0 \} \simeq GL(k)/U(k)$$

$$g^{*-1} g \ \longleftarrow\!\shortmid\ g$$

$\mathrm{Tr}\left(h^{-1}\frac{dh}{ds} \right)^2$ ... std Riem. metric on $\mathcal{H}$

$\underset{\geq 0}{2\,\mathrm{Tr}(\beta h^{-1}\beta^* h)}$ .... potential variational principle

$\Longrightarrow$

Prop. $\forall h_\pm \in \mathcal{H}$ $\exists h : \overset{[t_{-1}, t_+]}{\overset{\shortparallel}{I}} \to \mathcal{H}$ minimizing $\mathcal{L}$

s.t. $h(t_\pm) = h_\pm$

$\therefore$ $\exists g$ s.t. $\hat{F}(\alpha^g, \beta^g) = 0$ in I

○ uniqueness part $\Leftarrow$ convexity

$\Phi(h) := \log \max(\lambda_i \mid i = 1, \cdots, k)$

$\lambda_i =$ eigenvalues of h.

continuous func. on $\mathcal{H}$

Lemma (α, β), (α^g, β^g) : as before $\quad h := g^{*-1} g^{-1}$

$$\frac{d^2}{ds^2} \Phi(h) \geqq -2\left(|\hat{F}(\alpha, \beta)| + |\hat{F}(\alpha^g, \beta^g)|\right) \quad \text{in weak sense}$$

$$\frac{d^2}{ds^2} \Phi(h^{-1}) \geqq \quad \text{''}$$

proof) Assume h has distinct eigenvalues (everywhere)

May assume $g = \mathrm{diag}(e^{-t_1}, e^{-t_2}, \cdots, e^{-t_k})$ $\quad t_1 > t_2 > \cdots > t_k$

$h = \mathrm{diag}(e^{2t_1}, e^{2t_2}, \cdots \quad)$

$\therefore \Phi(h) = 2t_1$

$$g \hat{F}(\alpha^g, \beta^g) g^{-1} = \hat{F}(\alpha, \beta) - 2[\bar{\partial}_\alpha (h^{-1} \partial_\alpha h) + \bar{\partial}_\beta (h^{-1} \partial_\beta h)]$$

$$h^{-1} \partial_\alpha h = \mathrm{diag}\left(\frac{dt_1}{ds}, \cdots, \frac{dt_k}{ds}\right) + \alpha^* - h^{-1} \alpha^* h$$

$$\therefore\ \bar{\partial}_\alpha(h^{-1}\partial_\alpha h) = \frac{1}{2}\,\mathrm{diag}\left(\frac{d^2t_1}{ds^2}, \cdots, \frac{d^2t_k}{ds^2}\right) + \frac{1}{2}\frac{d}{ds}(\alpha^* - h^{-1}\alpha^* h)$$

$$+ \underline{\left[\alpha, \mathrm{diag}\left(\frac{dt_1}{ds}, \cdots \frac{dt_k}{ds}\right)\right]} + [\alpha, \alpha^* - h^{-1}\alpha^* h]$$

$$\bar{\partial}_\beta(h^{-1}\partial_\beta h) = [\beta, \beta^* - h^{-1}\beta^* h]$$

diag. comp. $= 0$

Take the $(1,1)$-component of $\left(\hat{F}(\alpha,\beta) - \underbrace{g}_{\text{diag}}\hat{F}(\alpha^g, \beta^g)\, g^{-1}\right)$:

$$\frac{d^2t_1}{ds^2} + 2\Big([\alpha, \underbrace{\alpha^* - h^{-1}\alpha^* h}_{\overline{\alpha_{ji}} - e^{-2t_j}\,\overline{\alpha_{ji}}\, e^{2t_i}}] + [\beta, \beta^* - h^{-1}\beta^* h]\Big)$$

$$\therefore (1,1)\text{-comp} = |\alpha_{1j}|^2\overbrace{(1 - e^{2(t_1 - t_j)})}^{<0} - |\alpha_{j1}|^2\underbrace{(1 - e^{2(t_j - t_1)})}_{>0}$$

$$< 0$$

$$\therefore\ \frac{d^2t_1}{ds^2} \geq \hat{F}(\alpha,\beta)_{(1,1)} - (g\hat{F}(\alpha^g, \beta^g)g^{-1})_{(1,1)}$$

$$= \left(\hat{F}(\alpha,\beta) - \hat{F}(\alpha^g, \beta^g)\right)_{(1,1)} \geq -\left(|\hat{F}(\alpha,\beta)| + |\hat{F}(\alpha^g,\beta^g)|\right)$$

$\Phi(h^{-1})$: similar //

Cor. The minimizer is unique up to unitary gauge transformations

☺ $\hat{F}(\alpha,\beta) = 0 = \hat{F}(\alpha^g, \beta^g)$

$\Longrightarrow \quad \dfrac{d^2}{ds^2}\Phi(h) \geq 0$ But $\Phi(h) = 0$ at $s = t_\pm$

maximum principle $\Rightarrow$ $\Phi(h) \leq 0$ i.e. all eigenvalues of $h \leq 1$

$\Phi(h^{-1}) \leqq 0$ i.e. all eigenvalues of $h \geqq 1$

$\therefore$ $h = 1$ //

Th. (Kronheimer)

$$\left\{\begin{array}{l}\text{solutions of Nahm's} \\ \text{equation up to unitary} \\ \text{gauge } g(t_I)=1\end{array}\right\} \cong T^*GL(k) = GL(k) \times gl(k)$$

(proof) LHS $\cong$ sol. of cpx equation up to cpx gauge $g(t_I)=1$

$$\left\{(\alpha,\beta) \,\middle|\, \frac{d}{ds}\beta + [\alpha,\beta]=0\right\} \Big/ \mathcal{G}^{\mathbb{C}}_{0*} \xrightarrow{\cong} gl(k)$$

constant solution

$$\therefore \left\{(\alpha,\beta) \,\middle|\, \frac{d}{ds} + [\alpha,\beta]=0\right\} \Big/ \mathcal{G}^{\mathbb{C}}_{00} \xrightarrow{\cong} GL(k) \times gl(k)$$ //

Rem. LHS is a hyperKähler manifold.